2.7 Linear Programming: Revision

Standard Form

Consider the LP problem in general with variable $x \in R^n$:

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & A_1 x = b_1 \\ & A_2 x \geq b_2 \\ & A_3 x \leq b_3 \\ & x_j \geq 0, \quad j \in G \\ & x_j \ \ unrestected, \quad j \in \bar{G} \end{aligned}$$

Before the simplex algorithm can be used to solve an LP, the LP must be converted into an equivalent problem in which *all constraints are equations and all variables are nonnegative*

An LP in this form is said to be in **standard form**.

• Eliminate unrestricted variables:

$$x_j = x_j^+ - x_j^-, \qquad \forall j \in \bar{G}, \qquad where\, x_j^+ \geq 0, \qquad x_j^- \geq 0.$$

• Rewrite constraints to guarantee:

$$b_i \geq 0, \qquad i = 1, \dots, m.$$

• Eliminate inequality constraints:

- For $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$, introduce surplus variable s_i and

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i$$

- For $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, introduce slack variables s_i and

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i$$

The simplex algorithm *starts* from a basic feasible solution on the boundary of the polyhedron representing the feasible region of the problem and hops to an adjacent basic feasible solution improving the value of the objective. The algorithm *stops* when no further improvement is possible. The algorithm also detects infeasibility (no value satisfies the constraints) and unboundedness where the objective can be made arbitrarily large.

Basic Solution

Given A is a $m \times n$ matrix. Assume A has m linearly independent columns, then these columns form a basis of A, which is denoted by $B = \{A_{j1}, A_{j2}, \dots, A_{jm}\}$. The solution corresponding to B is a vector $x \in R^n$ with

$$\begin{cases} x_j = 0, & A_j \notin B \\ x_k = the\ k-th\ compenent\ of\ B^{-1}b, & k = 1, \dots, m \end{cases}$$

Note that basic solution may be feasible or not. Simplex method however, works only with *basic feasible solutions*.

Procedure: (I-phase) Simplex Algorithm

```
begin (for max problem with z − Σ_j c_j x_j = 0 being the cost row)
    OPT := "NO"
    UNBOUNDED := "NO"
    while OPT = "NO" and UNBOUNDED = "NO", do
        if c_j ≥ 0, ∀j then OPT := "YES"
        else
            begin
                choose j such that c_j < 0 is the most negative
                if y_ij ≤ 0, ∀i then UNBOUNDED := "YES"
                else
                    find θ_0 = min{ y_i0 / y_ij , ∀i and y_ij ≥ 0}
                    denote the index by k and pivot on k
                endif
            end
        endif
    endwhile
end
```

Example:

Consider the LP problem

$$\begin{aligned} \max \quad & 8x_1 + 5x_2 \\ s.t \quad & x_1 + x_2 \le 6 \\ & 9x_1 + 5x_2 \le 45 \\ & x_1, \quad x_2 \ge 0 \end{aligned}$$

Solution:

Standard form

$$z - 8x_1 - 5x_2 = 0$$
$$x_1 + x_2 + s_1 = 6$$
$$9x_1 + 5x_2 + s_2 = 45$$
$$x_1, x_2, s_1, s_2 \geq 0$$

First bfs is z = 0, $x_1 = 0$, $x_2 = 0$, $s_1 = 6$, $s_2 = 45$

	Z	NBV x_1 ↓	NBV x_2	BV s_1	BV s_2	b	feasible ratio
	1	-8	-5	0	0	0	
	s_1	1	1	1	0	6	6/1=6
←	s_2	9	5	0	1	45	45/9=5
	1	0	-5/9	0	8/9	40	
←	s_1	0	4/9	1	-1/9	1	9/4= 2.5
	x_1	1	5/9	0	1/9	5	59/5=9
	1	0	0	5/4	3/4	165/4	
	x_2	0	1	9/4	-1/4	9/4	
	x_1	1	0	-5/4	1/4	15/4	

Then $x_1 = \frac{15}{4}$, $x_2 = \frac{9}{4}$, $z = \frac{165}{4}$

Remarks

- If the ratio test for the selected pivot column has no positive value, then we can increase the corresponding variable as much as we want, which makes the problem **unbounded**.
- For minimization problems consider

$$\min(z) = -\max(-z) = -\sum_j -c_j x_j$$

Recall that the simplex algorithm requires a starting basic feasible solution. In those problems that all constraints are $\leq$ constraints, we could start by using the slack variables as our basic variables as it implies $x_1 = x_2 = \cdots = x_n = 0$ and we start from the origin.

If an LP has any $\geq$ or equality constraints, however, a starting bfs may not be readily apparent. When a basic feasible solution is not readily available, we use the **Big M** method or the **two-phase** simplex algorithm may be applied.

- We add artificial variables to the $\geq$ or = constraints.
- Find a basic feasible solution to the original LP by solving the Phase I problem.
- In Phase I, the objective is to *minimize the sum of all artificial variables*.
- The act of solving the Phase I problem will force the artificial variables to be zero.
- At the completion of Phase I, we reintroduce the original LP's objective function and continue to Phase II to determine the optimal solution to the original LP.

Remarks (p172 winston)
If the optimal value of Phase I model comes out to be nonzero, the original LP is **infeasible**

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 \\ s.t \quad & \frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 4 \\ & x_1 + 3x_2 \geq 20 \\ & x_1 + x_2 = 10 \\ & x_1, \quad x_2 \geq 0 \end{aligned}$$

Put the system into standard form, add a slack variable s_1 to the first constraint and subtract an excess variable e_2 from second constraint. Write the o.f as $z - 2x_1 + 3x_2 = 0$. We got

Problem 1

$$\begin{aligned} \text{Row0:} \quad & z - 2x_1 - 3x_2 = 0 \\ Row1: \quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\ Row2: \quad & x_1 + 3x_2 - e_1 = 20 \\ Row3: \quad & x_1 + x_2 = 10 \end{aligned}$$

$$all\ variables\ are\ nonnegative$$

In searching for a bfs, we see that $s_1 = 4$ could be used as a bf variables in R1. In R2, multiply the row by -1, then $e_1 = -20$, we can use it as basic variable, but it should be $e_1 \geq 0$. In R3 there is no basic variable. So to use the

simplex algorithm we need to have basic variables in R2 and R3. So we add the artificial variables a_2, a_3.

Problem 2

$$\begin{aligned} z - 2x_1 + 3x_2 &= 0 \\ \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 &= 4 \\ x_1 + 3x_2 - e_1 + a_2 &= 20 \\ x_1 + x_2 + a_3 &= 10 \end{aligned}$$

We now have a bfs: $z = 0, s_1 = 4$, $a_2 = 20$, and $a_3 = 10$.
There is no guarantee that optimal solution of problem 2 will be the same to problem 1, we may obtain an optimal solution in which one or more artificial variables are positive. Such solution may not be feasible in the original solution. For example in problem 2 the optimal may the optimal is $z = 0, s_1 = 4$, $a_2 = 20$, and $a_3 = 10, x_1 = x_2 = 0$. Which deos not solve the original problem.

Because a_2 and a_3 are in our starting bfs (that's why we introduced them), they must be eliminated from row 0. To eliminate a_2 and a_3 from row 0.

We need to be sure that the artificail variables are equal to zero by adding a term Ma_i to the o.f for each artificail variable a_i in Min problem and add a term $-Ma_i$ to the Max problem. Here M is very large positive number, thus in problem 2 we change our o.f to

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

Then Row 0 will change to

$$z - 2x_1 + 3x_2 - Ma_2 - Ma_3 = 0$$

Modifying the o.f in this way makes is very costly for an artificial variable to be positive. So the optimal of problem 2 will have $a_2 = a_3 = 0$. Now the optimal of pronlem 2 will solve the original problem in problem 1.

Summary: sometimes, however, some of the artificial variables in problem 2 may assume positive values in the optimal solution. If this happens, the original problem has no feasible solution. The method above called the Big M method.

2.8 Linear Programming: Duality

Associated with any LP is another LP, called the **Dual**. When taking the dual of a given LP, we refer to the given LP as the **primal**.

Primal	**Dual**
$z_P^* = \max \quad c^T x$	$z_D^* = \min \quad b^T y$
$s.t. \quad Ax \le b$	$s.t. \quad A^T y \ge c$
$x \ge 0$	$y \ge 0$

The *i*th dual *constraint* corresponds to the *i*th primal *variable* x_i. In a similar fashion, dual *variable* y_i is associated with the *i*th primal constraint.

There are two ideas fundamental to duality theory:

- The dual of a dual LP is the original primal LP;
- Every feasible solution for a dual LP gives an upper bound on the optimal value of the objective function of its primal LP.

In linear programming problems, we always have **Strong Duality** holds:

$$z_P^* = \ c^T x^* = \ b^T y^* = z_D^*.$$

Example:

(P)
$$\begin{aligned} z_{LP} = \max \ & 7x_1 + 2x_2 \\ & -x_1 + 2x_2 \le 4 \\ & 5x_1 + x_2 \le 20 \\ & -2x_1 - 2x_2 \le -7 \\ & x_1, \quad x_2 \ge 0 \end{aligned}$$

is

(D)
$$\begin{aligned} w_{LP} = min \ & 4u_1 + 20u_2 - 7u_3 \\ & -u_1 + 5u_2 - 2u_3 \ge 7 \\ & 2u_1 + \ u_2 - 2u_3 \ge 2 \\ & u_1, \ u_2, \ u_3 \ge 0 \end{aligned}$$

The solution $x^* = \left(\frac{36}{11}, \frac{40}{11}\right)$ is feasible in P, and hence $z_{LP} \ge cx^* = 30\frac{2}{11}$.
Similarly, $u^* = \left(\frac{3}{11}, \frac{16}{11}, 0\right)$ is feasible in D, and hence $z_{LP} \le u^* b = 30\frac{2}{11}$.
The two points together yield a proof of optimality, x^* for P and u^* for D.

The Dual Simplex Algorithm

In the **(primal) Simplex Algorithm** we have discussed (we will refer to the max problem as a primal):

- We begin with a primal feasible solution as each constraint in the initial tableau has a nonnegative r.h.s.

- However, at least one variable in row 0 (objective row) has a negative coefficient, so our initial primal solution is not dual feasible .
- Through a sequence of simplex pivots, we maintain primal feasibility and obtain an optimal solution when dual feasibility (a nonnegative row 0) is attained.

In many situations, however, it is easier to solve an LP by beginning with a dual feasible tableau (in which each variable in row 0 has a nonnegative coefficient) but primal infeasible (at least one constraints has a negative r.h.s.), e.g.

when a new constraint is added to the LP.

The dual simplex method is often used to find the new optimal solution to an LP after a constraint is added. When a constraint is added, one of the following can be happened; the current optimal satisfies the new constraint, or the current optimal does not satisfy the new constraint, but the LP still has a feasible solution, or the new constraint makes the LP have no feasible solution.

The **Dual Simplex Algorithm** is then needed.

Procedure: Dual Simplex Algorithm
Step 1: Is the r.h.s. of each constraint nonnegative? If so, an optimal solution has been found and stop; if not, go to Step 2.

Step 2: Choose the *most negative* basic variable as the variable to leave the basis. To select the variable that enters the basis, we compute the following ratio for each variable x_j that has a *negative* coefficient in the pivot row:

$$\frac{\text{Coefficient of } x_j \text{ in row 0}}{\text{Coefficient of } x_j \text{ in pivot row}}$$

Choose the variable with the *smallest* ratio (absolute value) as the entering variable. Conduct elementary row operations to eliminate nonzero entries in the pivot column.

Step 3: If there is any constraint in which the r.h.s. is negative and each variable has a nonnegative coefficient, then the LP has *no* feasible solution. If no constraint indicating infeasibility is found, return to Step 1.